

INTEGRALITY OF HLK KERNELS

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ABSTRACT. We prove that the coefficients of the generating function of Hausel, Letellier, Villegas, and its recent generalization by Carlsson and Villegas, which according to various conjectures should compute mixed Hodge numbers of character varieties and moduli spaces of Higgs bundles of curves of genus g with n punctures, are polynomials in q and t with integer coefficients for any $g, n \geq 0$.

1. INTRODUCTION

Consider a Riemann surface of genus g with n punctures, $g, n \geq 0$. To formulate the main result and to fix the notations we introduce the *exponential HLK kernel* as follows:

$$\Omega_{u_1, u_2, \dots, u_g}[X_1, X_2, \dots, X_n; q, t, T] = \sum_{\lambda \in \mathcal{P}} \frac{\prod_{i=1}^n \tilde{H}_\lambda[X_i; q, t] \prod_{i=1}^g N_\lambda(u_i; q, t)}{(\tilde{H}_\lambda, \tilde{H}_\lambda)^S} T^{|\lambda|},$$

where

$$N_\lambda(u; q, t) = \prod_{s \in \lambda} (q^{a(s)} - ut^{l(s)+1})(q^{a(s)+1} - u^{-1}t^{l(s)}).$$

For the purpose of this introduction, X_i for each i denotes an infinite set of variables $X_{i1}, X_{i2}, X_{i3}, \dots$, and the resulting function is symmetric in each set. We find it more convenient to think about X_i as a formal symbol representing a generator of a free λ -ring, and this approach will be used throughout the paper. The summation runs over the set of all partitions, and for each partition λ we have a product of several terms. For each puncture we take the modified Macdonald polynomial $\tilde{H}_\lambda[X_i; q, t]$ of [GHT99a]. It is a symmetric function in X_{i1}, X_{i2}, \dots , whose coefficients are polynomials in q and t with integer coefficients. For each $i = 1, \dots, g$ we have an extra variable u_i , and an extra factor $N_\lambda(u_i; q, t)$, defined as a product over the cells of λ . It is a polynomial in q and t and a Laurent polynomial in u . The denominator is an appropriately modified Hall inner product, and can also be given explicitly by

$$(\tilde{H}_\lambda, \tilde{H}_\lambda)^S = N_\lambda(1; q, t) \in \mathbb{Z}[q, t].$$

Finally, the remaining term $T^{|\lambda|}$ is introduced to keep track of the degrees, and is necessary for convergence if $n = 0$.

The logarithmic HLV kernel is the unique series $\mathbb{H}_{u_1, u_2, \dots, u_g}[X_1, X_2, \dots, X_n; q, t, T]$ satisfying

$$(1) \quad \Omega_{u_1, u_2, \dots, u_g}[X_1, X_2, \dots, X_n; q, t, T] = \text{Exp} \left[-\frac{\mathbb{H}_{u_1, u_2, \dots, u_g}[X_1, X_2, \dots, X_n; q, t, T]}{(q-1)(t-1)} \right].$$

A traditional way to define Exp is to set

$$\text{Exp} \left[\sum_{i=1}^{\infty} c_i T_i \right] = \prod_{i=1}^{\infty} \frac{1}{(1 - T_i)^{c_i}}$$

for a sum of monomials T_i in some variables, and for constants c_i . Then to make sense of (1) we expand both sides as Laurent power series in q or t . A more canonical way to define Exp is in the setting of λ -rings, see below.

The conjectures, formulated in [HLRV11], [Moz12], [CRV16] and other works can be split into two parts.

Conjecture (Part 1). Denote by $\mathbb{H}_{u_1, u_2, \dots, u_g, \lambda}(q, t)$ the coefficient of \mathbb{H} in front of the monomial $\prod_{i=1}^n \prod_j X_{ij}^{\lambda_j^{(i)}}$ for a tuple of partitions $\lambda = (\lambda^{(1)}, \lambda^{(2)}, \dots, \lambda^{(n)})$. Then

$$\mathbb{H}_{u_1, u_2, \dots, u_g, \lambda}(q, t) \in \mathbb{Z}[q, t, u_1, \dots, u_g, u_1^{-1}, \dots, u_g^{-1}].$$

The second part interprets coefficients of these polynomials as some cohomological invariants associated to character varieties or moduli spaces of Higgs bundles. Define the mixed Hodge polynomial of a smooth affine variety X of dimension d by

$$\sum_{i=0}^d \sum_{j=0}^i q^{\frac{d-i}{2}} t^{\frac{j}{2}} \dim (W_{i+j} H^i(X, \mathbb{C}) / W_{i+j-1} H^i(X, \mathbb{C})) \in \mathbb{Z}[q^{\frac{1}{2}}, t^{\frac{1}{2}}],$$

where W denotes the weight filtration introduced by Deligne in [Del71]. In the case of character varieties we have ([HLRV11])

Conjecture (Part 2). Setting $u_1 = u_2 = \dots = u_g = -(qt)^{-\frac{1}{2}}$ in $\mathbb{H}_{u_1, u_2, \dots, u_g, \lambda}(q, t)$ we obtain the mixed Hodge polynomial of the character variety of a Riemann surface of genus g with n punctures and generic semi-simple conjugacy classes of type λ .

The main result of this paper is a proof of Part 1 (Corollary 7.2). There are two main techniques developed here. The first one centers around the notion of admissibility, similar to the one of [KS08]. We call an expression C admissible if it

is of the form

$$C = \text{Exp} \left[-\frac{L}{(q-1)(t-1)} \right],$$

where L is such that all of its coefficients are polynomial in q and t . We prove that admissibility is closed under the following operation. Suppose $C[X, Y]$ depends as a symmetric function on two sets of variables X, Y , and possibly on some other variables. Expand C as follows

$$C[X, Y] = \sum_i F_i[X] G_i[Y]$$

and write

$$\int_X^S C[X, X^*] := \sum_i (F_i, G_i)_S,$$

where $(F_i, G_i)_S$ is the modified Hall inner product satisfying

$$(p_\lambda, p_\mu)^S = \prod_i -(q^{\lambda_i} - 1)(t^{\lambda_i} - 1)(p_\lambda, p_\mu),$$

where (p_λ, p_μ) is the usual Hall inner product. We show that for an admissible C the result, if the corresponding infinite sums make sense, is also admissible. In fact, S above stands for $S = -(q-1)(t-1)$, and our result (see Lemma 5.3 and Theorems 5.2, 5.5) hold for a more general class of such S , which we call “good modifiers”, see Definition 5.1. The proof is constructive and gives a formula for L in $\int_X^S C[X, X^*] = \text{Exp} \left[-\frac{L}{(q-1)(t-1)} \right]$ as a certain sum over graphs, similar to the Feynman diagram decompositions in physics. This formula is established first for the unmodified Hall inner product in a simpler case (Theorem 3.4). Then we show how the arguments need to be modified in the general case (Theorem 4.1), and analyse how introduction of the modifier affects the result in Lemma 5.3.

The operation introduced above allows us to “build Ω ” using a single building block, namely our version of the operator ∇ of [BG99], [BGHT99], which is obtained from the original by a simple sign change (10). It is admissible in the sense that when we apply ∇ to the Cauchy kernel $\text{Exp} \left[-\frac{XY}{(q-1)(t-1)} \right]$, we get an admissible result. Admissibility of ∇ , which is our second main ingredient, is deduced from the one of the operator Δ_v , also introduced in [BGHT99]. We prove admissibility of Δ_v (Theorem 6.2) by showing that $L_v[X, Y]$ defined by

$$\Delta_v \text{Exp} \left[-\frac{XY}{(q-1)(t-1)} \right] = \text{Exp} \left[-\frac{L_v[X, Y]}{(q-1)(t-1)} \right]$$

can be computed by a certain recursion, which follows from a recently established identity satisfied by Δ_v (see [GM16]).

We finish this Introduction by quoting a simple special case of a fact, crucial in the proof of Lemma 5.3 and Theorems 5.2, 5.5, which may be interesting on its own. Let $\Gamma = (V, E)$ be a connected graph with Betti number $b(\Gamma) = 1 + \#E - \#V$. Let $m : V \cup E \rightarrow \mathbb{Z}_{>0}$ be a coloring of the vertices and edges of E by positive integers. Suppose $m(v) | m(e)$ for each edge $e \in E$ incident to a vertex $v \in V$. In the case when Γ is a tree we further assume that the g.c.d. of the numbers $m(e)$ ($e \in E$), $m(v)$ ($v \in V$) is 1. Then we have

$$(q-1) \frac{\prod_{e \in E} (q^{m(e)} - 1)}{\prod_{v \in V} (q^{m(v)} - 1)} \in (q-1)^{b(\Gamma)} \mathbb{Z}[q].$$

I thank Fernando Rodriguez Villegas and Erik Carlsson for useful discussions. This research was performed during my stay at SISSA, and partly at ICTP, Trieste, to which I am grateful for support and stimulating environment.

2. NOTATIONS

The ring of symmetric functions in infinitely many variables over \mathbb{Q} is denoted as Sym . A λ -ring is a ring R with an action of Sym

$$F[X] \quad (F \in \text{Sym}, X \in R),$$

such that the power sums p_n act by ring endomorphisms for all $n \in \mathbb{Z}_{>0}$ and

$$p_1[X] = X, \quad p_m[p_n[X]] = p_{mn}[X] \quad (X \in R, m, n \in \mathbb{Z}_{>0}).$$

Free λ -rings with generators X_1, X_2, \dots will be denoted as $\text{Sym}[X_1, X_2, \dots]$. Note that $\text{Sym}[X]$ and Sym is the same thing. We denote by $\text{Sym}[X_1, X_2, \dots]^*$ the ideal consisting of symmetric functions of positive degree, by $\text{Sym}[[X_1, X_2, \dots]]$ the completion of $\text{Sym}[X_1, X_2, \dots]$ with respect to this ideal, and by $\text{Sym}[[X_1, X_2, \dots]]^*$ the corresponding ideal in $\text{Sym}[[X_1, X_2, \dots]]$.

The set of partitions is denoted by \mathcal{P} . The set of non-empty partitions is denoted by \mathcal{P}^* . For any $\lambda \in \mathcal{P}$ we write

$$\lambda = (\lambda_1, \lambda_2, \dots, \lambda_{l(\lambda)}) \quad (\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_{l(\lambda)} > 0, l(\lambda) \geq 0).$$

The size of λ is

$$|\lambda| = \sum_{i=1}^{l(\lambda)} \lambda_i,$$

the set of all partitions of size n is

$$\mathcal{P}_n = \{\lambda \in \mathcal{P} : |\lambda| = n\}.$$

Denote

$$\text{Aut}(\lambda) = \{\pi \in S_{l(\lambda)} : \lambda_{\pi_i} = \lambda_i \text{ for all } 1 \leq i \leq l(\lambda)\}.$$

For a partition λ and a constant c we denote

$$c\lambda = (c\lambda_1, c\lambda_2, \dots, c\lambda_{l(\lambda)}).$$

In general, we will often assume our λ -ring Λ is complete with respect to some decreasing filtration $\Lambda = J^0 \supset J^1 \supset \dots$ satisfying $J^i J^{i'} \subset J^{i+i'}$ for each $i, i' \geq 0$. We usually do not specify the filtration, so that the exposition is less cluttered, and hope that it is obvious from the context. Tensor product of such rings means *completed tensor product*. We call a sequence $A_1, A_2, \dots \in \Lambda$ *well-behaved* if $p_\lambda[A_i]$ tends to 0 when $i, |\lambda|$ tend to ∞ . Equivalently, for each k there exists i_0 such that $p_\lambda[A_i] \in J^k$ if $i > i_0$ or $|\lambda| > i_0$. A single element X is well-behaved if the sequence $X, 0, 0, \dots$ is. A series is well-behaved if the sequence of its coefficients is.

We often have operators from $\Lambda \otimes \text{Sym}[X]$ to $\Lambda \otimes \text{Sym}[[X]]$. In such a situation we say that an operator is *finite* if its image is in $\Lambda \otimes \text{Sym}[X]$, and we say it is *continuous* if it extends to a continuous operator from $\Lambda \otimes \text{Sym}[[X]]$ to $\Lambda \otimes \text{Sym}[[X]]$. Note that two such operators U, V can be composed to form a new operator UV if V is finite or U is continuous. We then say that U and V are *composable*.

Plethystic exponential is the following expression, well-defined if X is well-behaved:

$$\text{Exp}[X] = \sum_{n=0}^{\infty} h_n[X] = \exp \left(\sum_{n=1}^{\infty} \frac{p_n[X]}{n} \right),$$

where h_n denotes the complete homogeneous symmetric function, and p_n denotes the power sum of degree n . The inverse operation is given as follows:

$$\text{Log}[1 + X] = \sum_{n=1}^{\infty} \frac{\mu(n)}{n} p_n[\log(1 + X)],$$

where μ is the Möbius function.

We also use e_n for the elementary symmetric function of degree n , m_λ for the monomial symmetric function, and set

$$p_\lambda = \prod_{i=1}^{l(\lambda)} p_{\lambda_i}, \quad h_\lambda = \prod_{i=1}^{l(\lambda)} h_{\lambda_i}, \quad e_\lambda = \prod_{i=1}^{l(\lambda)} e_{\lambda_i} \quad (\lambda \in \mathcal{P}).$$

3. CONVOLUTION FORMULA

We consider the free λ -ring $\Lambda_{A,B}$ over \mathbb{Q} with generators

$$\{A_\lambda, B_\lambda\}_{\lambda \in \mathcal{P}^*},$$

bi-graded by putting A_λ, B_λ in degrees $(|\lambda|, 0)$ and $(0, |\lambda|)$ respectively. We can form two series in the appropriate completion of $\Lambda_{A,B}[X]$, the free λ -ring generated by X and the generators of $\Lambda_{A,B}$:

$$(2) \quad A[X] := \sum_{\lambda \in \mathcal{P}^*} A_\lambda p_\lambda[X], \quad B[X] := \sum_{\lambda \in \mathcal{P}^*} B_\lambda p_\lambda[X],$$

and we are interested in the Hall scalar product of their plethystic exponentials with respect to the X variable

$$C(A, B) := (\text{Exp}[A[X]], \text{Exp}[B[X]])_X = \sum_{n=0}^{\infty} C_n(A, B),$$

where each $C_n(A, B) \in \Lambda_{A,B}$ is homogeneous of bi-degree (n, n) . The problem we are going to solve is to compute $L(A, B)$

$$L(A, B) = \sum_{n=1}^{\infty} L_n(A, B),$$

with each $L_n(A, B) \in \Lambda_{A,B}$ homogeneous of bi-degree (n, n) , such that

$$C(A, B) = \text{Exp}[L(A, B)].$$

This will clearly solve the problem of computing the Hall scalar product of arbitrary two well-behaved series of the form $\text{Exp}[\dots]$, and representing the result again in the same form.

3.1. Expansion of Exp into types.

Definition 3.1. A *type*¹ is an unordered tuple of pairs of the form (k, λ) , where $\lambda \in \mathcal{P}^*$, $k \in \mathbb{Z}_{>0}$. The set of all types is denoted by \mathcal{T} . More explicitly, choose a total ordering \leq on the set of all pairs (k, λ) as above. Then a type is given by a sequence

$$\tau = \{(k_i, \lambda^{(i)})\}_{i=1}^r \quad ((k_1, \lambda^{(1)}) \leq (k_2, \lambda^{(2)}) \leq \dots \leq (k_r, \lambda^{(r)}), \quad r \in \mathbb{Z}_{\geq 0}).$$

The degree of a type $\tau = \{(k_i, \lambda^{(i)})\}_{i=1}^r$ is the sum

$$|\tau| = \sum_{i=1}^r k_i |\lambda^{(i)}|.$$

¹We borrow the word “type” from [HLRV11], it seems to be an abbreviation of “ GL_n -type”. These classify combinatorial types of conjugacy classes and representations of GL_n over a finite field.

The set of all types of degree n is denoted by \mathcal{T}_n . The group of automorphisms of τ is defined as

$$\text{Aut}(\tau) = \{\pi \in S_r : k_{\pi_i} = k_i, \lambda^{(\pi_i)} = \lambda^{(i)} \text{ for all } 1 \leq i \leq r\}.$$

The partition $\text{flat}(\tau)$ is defined as the partition with components $k_i \lambda_j^{(i)}$, where i, j run over $i = 1, 2, \dots, r, j = 1, 2, \dots, l(\lambda^{(i)})$.

For any type $\tau = \{(k_i, \lambda^{(i)})\}_{i=1}^r$ and a collection of elements $Z = \{Z_\lambda\}_{\lambda \in \mathcal{P}^*}$ in some λ -ring we denote

$$Z^\tau = \prod_{i=1}^r p_{k_i} [Z_{\lambda^{(i)}}].$$

If X is a single element of a λ -ring, we set $X_\lambda = p_\lambda[X]$ and define X^τ as above. We have

$$X^\tau = \prod_{i=1}^r p_{k_i} [p_{\lambda^{(i)}}[X]] = p_{\text{flat}(\tau)}[X].$$

The element $A^\tau B^{\tau'}$ for $\tau, \tau' \in \mathcal{T}$ has bi-degree $(|\tau|, |\tau'|)$, and these elements form a basis of $\Lambda_{A,B}$. The expressions $L(A, B)$ and $C(A, B)$ can be expanded as follows:

$$C(A, B) = \sum_{\tau, \tau' \in \mathcal{T}} C_{\tau, \tau'} A^\tau B^{\tau'}, \quad L(A, B) = \sum_{\tau, \tau' \in \mathcal{T}} L_{\tau, \tau'} A^\tau B^{\tau'},$$

$$C_{\tau, \tau'}, L_{\tau, \tau'} \in \mathbb{Q}.$$

First we show how to expand Exp in the following

Proposition 3.1. *We have*

$$\text{Exp} \left[\sum_{\lambda \in \mathcal{P}} A_\lambda \right] = \sum_{\tau \in \mathcal{T}} g_\tau A^\tau,$$

where for each type $\tau = \{(k_i, \lambda^{(i)})\}_{i=1}^r$

$$g_\tau = \frac{1}{\# \text{Aut}(\tau) \prod_{i=1}^r k_i}.$$

Proof. Simply use the expansion of Exp into the power sums. \square

We compute $C_{\tau, \tau'}$ first.

Proposition 3.2. *Let $\tau = \{(k_i, \lambda^{(i)})\}_{i=1}^r$, $\tau' = \{(k'_i, \lambda'^{(i)})\}_{i=1}^{r'}$ be types with $|\tau| = |\tau'|$. The number $C_{\tau, \tau'}$ is given by*

$$C_{\tau, \tau'} = \begin{cases} \frac{\# \text{Aut}(\text{flat}(\tau)) \prod_{i=1}^r \prod_{j=1}^{l(\lambda^{(i)})} k_i \lambda_j^{(i)}}{\# \text{Aut}(\tau) \# \text{Aut}(\tau') \prod_{i=1}^r k_i \prod_{i=1}^{r'} k'_i} & \text{if } \text{flat}(\tau) = \text{flat}(\tau'), \\ 0 & \text{otherwise.} \end{cases}$$

Proof. Using Proposition 3.1 we write

$$\text{Exp}[A[X]] = \sum_{\tau \in \mathcal{T}} g_{\tau} A^{\tau} p_{\text{flat}(\tau)}[X],$$

and similarly for $B[X]$. Then use the formula

$$(p_{\lambda}, p_{\lambda'}) = \begin{cases} \# \text{Aut}(\lambda) \prod_{i=1}^{l(\lambda)} \lambda_i & \text{if } \lambda = \lambda', \\ 0 & \text{otherwise.} \end{cases}$$

□

3.2. Graph interpretation. Next we note that $\# \text{Aut}(\text{flat}(\tau))$ in the case $\text{flat}(\tau) = \text{flat}(\tau')$ is the number of bijections φ from the set

$$W = \{(i, j) : 1 \leq i \leq r, 1 \leq j \leq l(\lambda^{(i)})\}$$

to the set

$$W' = \{(i, j) : 1 \leq i \leq r', 1 \leq j \leq l(\lambda'^{(i)})\},$$

such that $k_i \lambda_j^{(i)} = k'_{i'} \lambda'^{(i')}_j$ whenever $\varphi(i, j) = (i', j')$. We represent such a bijection by a bipartite graph with multi-edges as follows. The vertex set is the disjoint union of two sets $V = V_1 \sqcup V_2$ where $V_1 = \{1, 2, \dots, r\}$ and $V_2 = \{1, 2, \dots, r'\}$, i.e. vertices correspond to the pairs $(k_i, \lambda^{(i)})$ in τ and τ' . Vertices are colored by positive integers. The colors of $i \in V$ resp. $i' \in V'$ are given by k_i resp. $k'_{i'}$. The edges are also colored. We connect $i \in V$ to $i' \in V'$ by an edge with color $k_i \lambda_j^{(i)}$ for each pair j, j' such that $\varphi(i, j) = (i', j')$. The result is a bipartite graph with multi-edges (V, E, m) , where $m(v)$ resp. $m(e)$ denotes the color of vertex $v \in V$ resp. edge $e \in E$. The graph and the coloring satisfy the following condition:

Definition 3.2. A colored bipartite graph (V, E, m) with coloring $m : V \cup E \rightarrow \mathbb{Z}_{>0}$ is called *admissible* if for each edge e adjacent to a vertex v we have $m(v) | m(e)$.

Note that in our construction all admissible graphs without isolated vertices appear. From an admissible bipartite graph without isolated vertices we recover τ , τ' and the bijection φ as follows. To each $v \in V_1$ we associate a pair $(m(v), \lambda^{(v)})$, where $\lambda^{(v)} = \frac{1}{m(v)} \lambda_E^{(v)}$, and $\lambda_E^{(v)}$ is the partition with parts $m(e)$, where e runs over all edges adjacent to v . The collection of these pairs forms τ . Similarly we construct τ' . To define the bijection φ we use the edges. There is an ambiguity we need to fix. Each edge e connecting i and i' creates a part $\frac{m(e)}{m(i)}$ for the partition corresponding to i in τ , and a part $\frac{m(e)}{m(i')}$ for the partition corresponding to i' in τ' , and φ should associate the two parts. When we have several copies of the same part in $\lambda^{(i)}$, we

have to further choose which of these parts will be associated to which part at the other end of e . We notice that the product of the groups

$$\text{Aut}(V) := \prod_{i \in V_1} \text{Aut}(\lambda^{(i)}) \times \prod_{i \in V_2} \text{Aut}(\lambda'^{(i)}) = \prod_{v \in V} \text{Aut}(\lambda^{(v)})$$

transitively acts on all these choices with the stabilizer

$$\text{Aut}(E) := \prod_{i \in V_1, i' \in V_2} \text{Aut}(\lambda^{(i, i')}) \subset \text{Aut}(V),$$

where $\lambda^{(i, i')} \in \mathcal{P}$ is the partition consisting of the colors of all the edges between i and i' . Thus each graph corresponds to $\frac{\# \text{Aut}(V)}{\# \text{Aut}(E)}$ bijections φ . Hence we obtain

Proposition 3.3. *The number $C_{\tau, \tau'}$ is given as:*

$$C_{\tau, \tau'} = \sum_{E, m} \frac{\# \text{Aut}(V) \prod_{e \in E} m(e)}{\# \text{Aut}(E) \# \text{Aut}(\tau) \# \text{Aut}(\tau') \prod_{v \in V} m(v)},$$

where the sum is running over all E, m producing the pairs $(k_i, \lambda^{(i)})$ resp. $(k'_i, \lambda'^{(i)})$ at vertices $i \in V_1$ resp. $i \in V_2$.

3.3. Connected components decomposition. For any bipartite admissible graph $\Gamma = (V, E, m)$ with $V = V_1 \sqcup V_2$ define its *weight* as

$$w(\Gamma) = w(\Gamma)(A, B) = \frac{\# \text{Aut}(V) \prod_{e \in E} m(e)}{\prod_{v \in V} m(v)} A^\tau B^{\tau'} \in \Lambda_{A, B}.$$

It can also be written as

$$(3) \quad w(\Gamma) = \frac{1}{\prod_{x \in V \cup E} m(x)} \prod_{v \in V} \left(p_{m(v)}[A[X] \text{ or } B[X]], p_{\lambda_E^{(v)}}[X] \right)_X,$$

where we take $A[X]$ for $v \in V_1$ and $B[X]$ for $v \in V_2$. The weight is multiplicative:

$$w(\Gamma_1 \sqcup \Gamma_2) = w(\Gamma_1)w(\Gamma_2),$$

and we have

$$C(A, B) = \sum_{r, r'=0}^{\infty} \sum_{\substack{\Gamma=(V, E, m) \text{ admissible,} \\ V_1=\{1, \dots, r\}, V_2=\{1, \dots, r'\}}} \frac{w(\Gamma)}{\# \text{Aut}(E) r! r'}.$$

The last identity is true because in the right hand side each pair of types τ, τ' appears $\frac{r! r'!}{\# \text{Aut}(\tau) \# \text{Aut}(\tau')}$ times.

Finally, we forget the identifications $V_1 = \{1, \dots, r\}$, $V_2 = \{1, \dots, r'\}$ and obtain

$$(4) \quad C(A, B) = \sum_{\Gamma \text{ admissible}} \frac{w(\Gamma)}{\# \text{Aut}(\Gamma)}.$$

In this formula the automorphisms are understood as permutations of vertices and edges, preserving the incidence relation, the marking and the decomposition $V = V_1 \sqcup V_2$.

Each graph Γ is a union of its connected components in a unique way. Let Γ be the union of m_i copies of Γ_i for $i = 1, 2, \dots, k$. The automorphism group of Γ is the product

$$\text{Aut}(\Gamma) = \prod_{i=1}^k \text{Aut}(\Gamma_i)^{m_i} \rtimes S_{m_i}.$$

This leads to

$$(5) \quad C(A, B) = \exp \left(\sum_{\substack{\Gamma \text{ admissible,} \\ \text{connected}}} \frac{w(\Gamma)}{\# \text{Aut}(\Gamma)} \right).$$

3.4. Scaling operations. Finally, we analyse what happens when we apply the following *scaling* operation. For each $n \in \mathbb{Z}_{>0}$ and Γ as above let $p_n(\Gamma)$ be the graph with V, E the same as in Γ , and all values of m multiplied by n . For a connected graph Γ we have

$$(6) \quad w(p_n(\Gamma)) = p_n[w(\Gamma)] n^{b(\Gamma)-1},$$

where $b(\Gamma) = \#E - \#V + 1 \geq 0$ is the first Betti number of Γ . We call a graph *primitive* if it cannot be obtained as $p_n[\Gamma]$ for $n > 1$, i.e. if the g.c.d. of all the values of m is 1. Then every graph is expressed as $p_n(\Gamma)$ for a unique n and a primitive graph Γ . Thus we have

$$\log C(A, B) = \sum_{\substack{\Gamma \text{ admissible,} \\ \text{connected, primitive}}} \sum_{n=1}^{\infty} n^{b(\Gamma)-1} \frac{p_n[w(\Gamma)]}{\# \text{Aut}(\Gamma)}.$$

The Möebius inversion formula for Log, the operation inverse to Exp, is:

$$\text{Log}[X] = \sum_{n=1}^{\infty} \frac{\mu(n)}{n} p_n[\log(X)].$$

In our case this leads to

$$L(A, B) = \text{Log } C(A, B) = \sum_{\substack{\Gamma \text{ admissible,} \\ \text{connected, primitive}}} \sum_{n=1}^{\infty} n^{b(\Gamma)-1} \prod_{p|n \text{ prime}} (1 - p^{-b(\Gamma)}) \frac{p_n[w(\Gamma)]}{\# \text{Aut}(\Gamma)}.$$

Finally, we pass back to a sum over all connected graphs using (6):

Theorem 3.4 (Logarithmic convolution). *We have*

$$(7) \quad L(A, B) = \sum_{\substack{\Gamma \text{ bipartite,} \\ \text{admissible,} \\ \text{connected}}} \phi_{b(\Gamma)}(g(\Gamma)) \frac{w(\Gamma)(A, B)}{\# \text{Aut}(\Gamma)},$$

where $g(\Gamma)$ denotes the g.c.d. of all the values of the coloring, and

$$\phi_k(n) = \prod_{p|n \text{ prime}} (1 - p^{-k}).$$

Remark 3.1. Note that for $k = 1$ and $n > 1$ we have $\phi_k(n) = 0$. Hence terms with $g(\Gamma) > 1$ disappear when Γ is a tree. In what follows we will ignore graphs which are trees with $g(\Gamma) > 1$.

3.5. Examples. We compute first few terms in the expansion (7). Define the *degree* of an admissible graph as the sum.

$$|\Gamma| := \sum_{e \in E} m(e),$$

The simplest case is

3.5.1. Degree 0. Strictly speaking, we don't have admissible graphs of degree 0, but it is convenient to include the following two graphs in the formula (7)²:

$$\Gamma_1 = \bullet, \quad \Gamma_2 = \circ.$$

Now if we add to $\Lambda_{A,B}$ two more generators A_0, B_0 of degree 0, and set $w(\Gamma_1)(A, B) = A_0$, $w(\Gamma_2)(A, B) = B_0$, the formula (7) will still hold. So we have

$$L(A, B) = A_0 + B_0 + \dots$$

3.5.2. Degree 1. We have only 1 graph which is connected and has exactly one edge:

$$\Gamma = \bullet \text{---} \circ$$

This graph has weight $w(\Gamma) = A_{(1)}B_{(1)}$.

3.5.3. Degree 2. In degree 2 we have the following graphs:

$$\begin{aligned} \Gamma_1 &= \bullet \text{---}^2 \text{---} \circ, & \Gamma_2 &= \bullet \text{---}^2 \text{---} \circ^2, & \Gamma_3 &= \bullet^2 \text{---}^2 \text{---} \circ, \\ \Gamma_4 &= \bullet \text{---} \circ, & \Gamma_5 &= \bullet \text{---} \circ, & \Gamma_6 &= \bullet \text{---} \circ. \end{aligned}$$

²Drawing bipartite graphs we paint the vertices from V_1 in black and the ones from V_2 in white. We omit the values of m when they are 1

Their weights are given as follows:

$$\begin{aligned} w(\Gamma_1) &= 2A_{(2)}B_{(2)}, \quad w(\Gamma_2) = A_{(2)}p_2[B_{(1)}], \quad w(\Gamma_3) = p_2[A_{(1)}]B_{(2)}, \\ w(\Gamma_4) &= 4A_{(1,1)}B_{(1,1)}, \quad w(\Gamma_5) = 2A_{(1,1)}B_{(1)}^2, \quad w(\Gamma_6) = 2A_{(1)}^2B_{(1,1)}. \end{aligned}$$

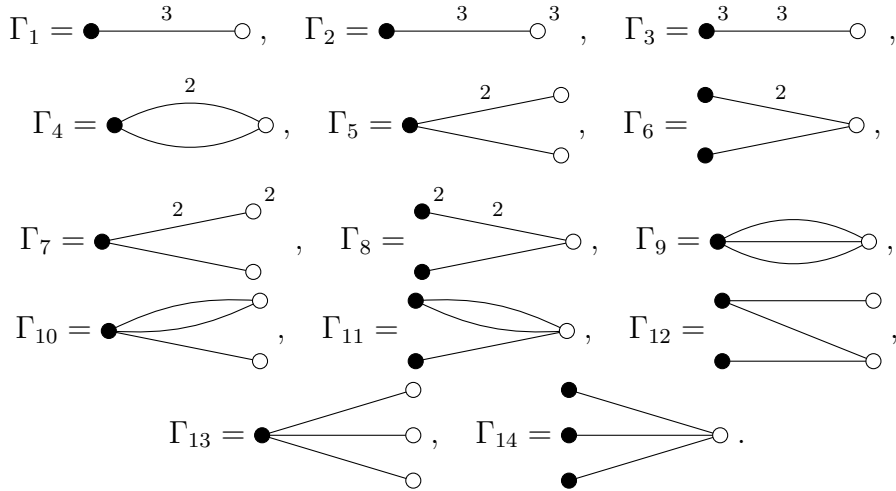
The orders of the automorphism groups are

$$\# \text{Aut}(\Gamma_i) = 1 \quad (i = 1, 2, 3), \quad \# \text{Aut}(\Gamma_i) = 2 \quad (i = 4, 5, 6).$$

So we obtain

$$\begin{aligned} L(A, B) &= A_{(0)} + B_{(0)} + A_{(1)}B_{(1)} + 2A_{(2)}B_{(2)} + A_{(2)}p_2[B_{(1)}] \\ &+ p_2[A_{(1)}]B_{(2)} + 2A_{(1,1)}B_{(1,1)} + A_{(1,1)}B_{(1)}^2 + A_{(1)}^2B_{(1,1)} + \cdots \end{aligned}$$

3.5.4. *Degree 3.* In degree 3 we have the following graphs:



These graphs produce 14 terms

$$\begin{aligned} &3A_{(3)}B_{(3)} + A_{(3)}p_3[B_{(1)}] + p_3[A_{(1)}]B_{(3)} \\ &+ 2A_{(2,1)}B_{(2,1)} + 2A_{(2,1)}B_{(2)}B_{(1)} + 2A_{(2)}A_{(1)}B_{(2,1)} \\ &+ A_{(2,1)}p_2[B_{(1)}]B_{(1)} + p_2[A_{(1)}]A_{(1)}B_{(2,1)} + 6A_{(1,1,1)}B_{(1,1,1)} \\ &+ 6A_{(1,1,1)}B_{(1,1)}B_{(1)} + 6B_{(1,1,1)}A_{(1,1)}A_{(1)} + 4A_{(1,1)}A_{(1)}B_{(1,1)}B_{(1)} \\ &+ A_{(1,1,1)}B_{(1)}^3 + A_{(1)}^3B_{(1,1,1)}. \end{aligned}$$

4. CONVOLUTION FORMULA II

To account for genus terms later we need a more general operation than the convolution of the previous Section. Consider the free λ -ring Λ_A over \mathbb{Q} with generators

$$\{A_{\lambda,\mu}\}_{\lambda,\mu \in \mathcal{P}}.$$

Define

$$A[X, X^*] = \sum_{\lambda, \mu \in \mathcal{P}} A_{\lambda, \mu} p_{\lambda}[X] p_{\mu}[X^*].$$

Define the exponential convolution as

$$C(A) := \int_X \text{Exp}[A[X, X^*]],$$

where \int_X is the Hall inner product map, viewed as a linear map

$$\text{Sym}[X, X^*] \cong \text{Sym}[X] \otimes \text{Sym}[X] \rightarrow \mathbb{Q}.$$

In other words, it is the linear map $\Lambda \otimes \text{Sym}[X, X^*] \rightarrow \Lambda$ satisfying

$$\int_X F[X] G[X^*] = (F[X], G[X])_X.$$

Similarly to the previous Section, we define $L[A]$ by

$$C(A) = \text{Exp}[L(A)],$$

and we would like to compute $L(A)$ as a sum over graphs. It turns out that all the constructions of the previous Section go through, and the result is essentially the same, except that the graphs we obtain are directed graphs with multiedges and loops. We go through the key points of the previous Section to highlight the differences.

- The constant term $A_{(), ()}$ produces just a factor $\text{Exp}[A_{(), ()}]$ in front of $C(A)$, therefore it only adds $A_{(), ()}$ to the result $L(A)$. So we reduce to the case $A_{(), ()} = 0$. We can treat $A_{(), ()}$ as the contribution of the graph with one vertex and no edges.
- Instead of pairs (d, λ) we have triples (d, λ, μ) with $d \in \mathbb{Z}_{>0}$, $\lambda, \mu \in \mathcal{P}$ such that $(\lambda, \mu) \neq ((), ())$. We choose a total ordering on such triples and define *double types* as sequences

$$\tau = \{(k_i, \lambda^{(i)}, \mu^{(i)})\}_{i=1}^r :$$

$$((k_1, \lambda^{(1)}, \mu^{(1)}) \leq (k_2, \lambda^{(2)}, \mu^{(2)}) \leq \dots \leq (k_r, \lambda^{(r)}, \mu^{(r)}), \quad r \in \mathbb{Z}_{\geq 0}.$$

- For each double type as above we have two partitions $\text{flat}_1(\tau)$, $\text{flat}_2(\tau)$, obtained by taking all the parts of $k_i \lambda^{(i)}$ ($i = 1, 2, \dots, r$), $k_i \mu^{(i)}$ ($i = 1, 2, \dots, r$) respectively.
- The notation A^τ means

$$A^\tau = \prod_{i=1}^r p_{k_i}[A_{\lambda^{(i)}, \mu^{(i)}}],$$

and

$$X^\tau = \prod_{i=1}^r p_{k_i}[p_{\lambda^{(i)}}[X]]p_{k_i}[p_{\mu^{(i)}}[X^*]] = p_{\text{flat}_1(\tau)}[X]p_{\text{flat}_2(\tau)}[X^*],$$

and an analog of Proposition 3.1 holds.

- Analog of Proposition 3.2 holds with

$$C_\tau = \begin{cases} \frac{\# \text{Aut}(\text{flat}_1(\tau)) \prod_{i=1}^r \prod_{j=1}^{l(\lambda^{(i)})} k_i \lambda_j^{(i)}}{\# \text{Aut}(\tau) \prod_{i=1}^r k_i} & \text{if } \text{flat}_1(\tau) = \text{flat}_2(\tau), \\ 0 & \text{otherwise.} \end{cases}$$

- The bijections φ are from the set

$$W = \{(i, j) : 1 \leq i \leq r, 1 \leq j \leq l(\lambda^{(i)})\}$$

to the set

$$W' = \{(i, j) : 1 \leq i \leq r, 1 \leq j \leq l(\mu^{(i)})\}.$$

- Instead of a colored bipartite graph we construct a colored directed graph with vertex set $V = \{1, 2, \dots, r\}$. Multiple edges and loops are allowed. The admissibility condition is the same (see Definition 4.1).
- For each vertex v we associate two partitions: $\lambda^{(v)} = \frac{1}{m(v)}\lambda_E^{(v)}$, $\mu^{(v)} = \frac{1}{m(v)}\mu_E^{(v)}$, where $\lambda_E^{(v)}$ resp. $\mu_E^{(v)}$ contains all the colors of the outgoing resp. incoming edges of a vertex v . The triples $(m(v), \lambda^{(v)}, \mu^{(v)})$ form the type associated to Γ . For each pair $v, v' \in V$ the partition $\lambda^{(v, v')}$ contains colors of all the edges from v to v' .
- We define

$$\text{Aut}(V) = \prod_{v \in V} \text{Aut}(\lambda^{(v)}) \times \text{Aut}(\mu^{(v)}),$$

$$\text{Aut}(E) = \prod_{v, v' \in V} \text{Aut}(\lambda^{(v, v')}) \subset \text{Aut}(V).$$

Then an analog of Proposition 3.3 holds:

$$C_\tau = \sum_{E, m} \frac{\# \text{Aut}(V) \prod_{e \in E} m(e)}{\# \text{Aut}(E) \# \text{Aut}(\tau) \prod_{v \in V} m(v)}.$$

- The weight is defined as

$$w(\Gamma) = w(\Gamma)(A) = \frac{\# \text{Aut}(V) \prod_{e \in E} m(e)}{\prod_{v \in V} m(v)} \prod_{v \in V} p_{m(v)}[A_{\lambda^{(v)}, \mu^{(v)}}],$$

and we have

$$C(A) = \sum_{\Gamma \text{ admissible}} \frac{w(\Gamma)}{\# \text{Aut}(\Gamma)},$$

where the automorphism group acts on edges and vertices, preserving the labels, the incidence relation, and the directions of edges.

- Discussions in Section 3.4 go unchanged.

To summarize, we have

Definition 4.1. A colored directed graph (V, E, m) with coloring $m : V \cup E \rightarrow \mathbb{Z}_{>0}$ is called *admissible* if for each edge e adjacent to a vertex v we have $m(v) | m(e)$.

Theorem 4.1 (Logarithmic convolution II). *We have*

$$(8) \quad L(A) = \sum_{\substack{\Gamma \text{ directed,} \\ \text{admissible,} \\ \text{connected}}} \phi_{b(\Gamma)}(g(\Gamma)) \frac{w(\Gamma)(A)}{\# \text{Aut}(\Gamma)},$$

where $g(\Gamma)$ denotes the g.c.d. of all the values of the coloring, and

$$\phi_k(n) = \prod_{p|n \text{ prime}} (1 - p^{-k}).$$

Example 4.1. Let $A[X, X^*] = AX X^*$ over the free λ -ring generated by A . We can use the Theorem with $A_{(1),(1)} = A$ and all other terms 0. Therefore, in the sum over graphs we only have connected graphs such that each vertex v has exactly one outgoing and one incoming edge. Moreover, the colors of the edges and the vertices must be all equal, let's say $m(e) = m(v) = m \in \mathbb{Z}_{>0}$. Such a graph is necessarily a cycle, say on n vertices with $n = 1, 2, 3, \dots$. We obtain

$$\text{Log} \int_X \text{Exp}[AX X^*] = \sum_{k, m \geq 1} \phi_1(m) \frac{p_m[A]^n}{n}.$$

For instance, if $A = u$ is a monomial, i.e. $p_m[u] = u^m$ for all m , we have

$$\text{Log} \int_X \text{Exp}[uX X^*] = \sum_{k, m=1}^{\infty} \frac{\varphi(m)}{mn} u^{mn},$$

where $\varphi(m)$ denotes the Euler's totient function. The last expression evaluates to

$$\text{Log} \int_X \text{Exp}[uX X^*] = \sum_{n=1}^{\infty} u^n = \frac{u}{1-u}.$$

5. MODIFIERS

We analyze how introduction of certain modifiers in the Hall product influences (7) and (8).

Definition 5.1. Let Λ be a λ -ring. We call an element $S \in \Lambda$ a *good modifier* if the following two properties hold:

- (i) For each $n \in \mathbb{Z}_{>0}$ the element $p_n[S]$ is not a zero divisor.
- (ii) For each $n \in \mathbb{Z}_{>0}$ we have $S|p_n[S]$.
- (iii) For each relatively prime pair $m, n \in \mathbb{Z}_{>0}$ we have $p_m[S]p_n[S] \mid Sp_{mn}[S]$.

Note that (ii) implies $p_m[S]|p_n[S]$ whenever $m|n$ because p_m is an algebra homomorphism. Typical modifiers are $S = q - 1$, $S = 1 - q$ for $\Lambda = \mathbb{Q}[q]$, and $S = -(1 - q)(1 - t)$ for $\Lambda = \mathbb{Q}[q, t]$.

Given a good modifier S we introduce the *modified Hall inner product* on $\Lambda \otimes \text{Sym}[X]$ as follows:

$$(F[X], G[X])_X^S = (F[SX], G[X])_X \quad (F, G \in \Lambda \otimes \text{Sym}[X]).$$

Note that because of the following identity the modified inner product is symmetric:

$$(F[SX], G[X])_X = (F[X], G[SX])_X$$

Denote by Λ_S the localization

$$\Lambda_S = \Lambda[S^{-1}, p_2[S]^{-1}, p_3[S]^{-1}, \dots].$$

The reproducing kernel of the identity operator with respect to the modified inner product is

$$\text{Exp} \left[\frac{XY}{S} \right] \in \Lambda_S \otimes \text{Sym}[[X, Y]],$$

i.e. the following identity holds

$$\left(\text{Exp} \left[\frac{XY}{S} \right], F[Y] \right)_Y^S = F[X] \quad (F \in \Lambda \otimes \text{Sym}[X]).$$

For any Λ -linear operator $U : \Lambda \otimes \text{Sym}[X] \rightarrow \Lambda_S \otimes \text{Sym}[[X]]$ we define its kernel as

$$K_U^S[X, Y] = U \text{Exp} \left[\frac{XY}{S} \right] \in \Lambda_S \otimes \text{Sym}[[X, Y]].$$

Then we have

$$(K_U^S[X, Y], F[Y])_Y^S = (UF)[X] \quad (F \in \Lambda \otimes \text{Sym}[X]).$$

For any two composable operators U, V we have

$$(9) \quad K_{UV}^S[X, Y] = (K_U^S[X, Z], K_V^S[Z, Y])_Z^S.$$

Now we define an important class of operators:

Definition 5.2. In general, an expression $A \in \Lambda_S$ is called *S-admissible* over Λ if it is of the form $\text{Exp} \left[\frac{L}{S} \right]$ for $L \in \Lambda$. An operator $U : \Lambda \otimes \text{Sym}[X] \rightarrow \Lambda_S \otimes \text{Sym}[[X]]$ is called *S-admissible* over Λ if its kernel has the form

$$K_U^S[X, Y] = \text{Exp} \left[\frac{L_U^S[X, Y]}{S} \right] \text{ with } L_U^S[X, Y] \in \Lambda \otimes \text{Sym}[[X, Y]],$$

where $L_U^S[0, 0]$, i.e. the constant term of $L_U^S[X, Y]$ is well-behaved.

It turns out that sometimes strong admissibility implies *integrality*:

Proposition 5.1. *If U is an S-admissible operator such that $U(1) \in \Lambda[[X]]$, then for any $F \in \Lambda \otimes \text{Sym}[X]$ we have*

$$UF \in \Lambda \otimes \text{Sym}[[X]].$$

Proof. We have

$$(UF)[X] = \left(\text{Exp} \left[\frac{L_U^S[X, SY]}{S} \right], F[Y] \right)_Y.$$

Write $L_U^S[X, Y] = L_0[X] + L_+[X, Y]$ where $L_0 = L_U^S[X, 0]$ and $L_+[X, Y]$ contains only terms of positive degree in Y . Then

$$\frac{L_+[X, SY]}{S} \in \Lambda \otimes \text{Sym}[[X, Y]], \text{ and } U(1) = \text{Exp} \left[\frac{L_0[X]}{S} \right] \in \Lambda \otimes \text{Sym}[[X]]$$

by the assumptions. Hence

$$\text{Exp} \left[\frac{L_U^S[X, SY]}{S} \right] \in \Lambda \otimes \text{Sym}[[X, Y]],$$

which implies integrality of UF . □

Our main result about admissible operators is

Theorem 5.2. *For any S-admissible composable operators U and V the composition UV is also admissible.*

The proof follows from (9) and the following fact after replacing Λ by $\Lambda[[X, Y]]$.

Lemma 5.3. *For any λ -ring Λ with a good modifier S , and for well-behaved $A, B \in \Lambda \otimes \text{Sym}[[X]]$ we have*

$$\left(\text{Exp} \left[\frac{A[X]}{S} \right], \text{Exp} \left[\frac{B[X]}{S} \right] \right)_X^S = \text{Exp} \left[\frac{L^S(A, B)}{S} \right] \quad \text{with } L^S(A, B) \in \Lambda.$$

Proof. It is enough to prove the statement for $\Lambda_{A,B}$, the free λ -ring over Λ generated by $\{A_\lambda, B_\lambda\}_{\lambda \in \mathcal{P}}$, as in Section 3, and $A[X], B[X]$ as in (2). See also Section 3.5.1 for the way to incorporate constant terms. Denote

$$A'_\lambda = \frac{A_\lambda p_\lambda[S]}{S}, \quad B'_\lambda = \frac{B_\lambda}{S}.$$

To compute C we apply Theorem 3.4 for A', B' :

$$\frac{C}{S} = \sum_{\substack{\Gamma \text{ admissible,} \\ \text{connected}}} \sum_{n=1}^{\infty} \phi_{b(\Gamma)}(g(\Gamma)) \frac{w(\Gamma)(A', B')}{\# \text{Aut}(\Gamma)}.$$

It is enough to show that for each graph Γ such that $\phi_{b(\Gamma)}(g(\Gamma)) \neq 0$ we have

$$S w(\Gamma)(A', B') \in \Lambda_{A,B}.$$

By the definition of $w(\Gamma)$, $S w(\Gamma)(A', B')$ is a rational multiple of

$$S \frac{S^\tau}{\prod_{v \in V} p_{m(v)}[S]} A^\tau B^{\tau'}.$$

Therefore, it is enough to show that the following expression is in Λ :

$$c_\Gamma[S] := S \frac{S^\tau}{\prod_{v \in V} p_{m(v)}[S]} = S \frac{\prod_{e \in E} p_{m(e)}[S]}{\prod_{v \in V} p_{m(v)}[S]}.$$

We show it by induction on the number of edges. We have two cases: either Γ is a primitive tree, or Γ is not a tree.

Suppose Γ is a primitive tree. Then removing a leaf vertex v adjacent to an edge e we obtain a tree Γ' , which is not necessarily primitive. Denote $g = g(\Gamma')$ and let $\Gamma' = p_g[\Gamma'']$, so that Γ'' is primitive. Then we have

$$c_{\Gamma''}[S] \in \Lambda, \quad c_\Gamma[S] = S \frac{p_{m(e)}[S]}{p_{m(v)}[S]} p_g \left[\frac{c_{\Gamma''}[S]}{S} \right] = \frac{S p_{m(e)}[S]}{p_{m(v)}[S] p_g[S]} p_g[c_{\Gamma''}[S]].$$

We have $\gcd(g, m(v)) = 1$ because Γ is primitive, and $g|m(e)$, $m(v)|m(e)$ because Γ is admissible. Therefore $gm(v)|m(e)$ and the statement follows from the axioms (ii) and (iii) of Definition 5.1.

Now suppose that Γ has a cycle. Then it has an edge e such that removing e does not destroy connectivity. Let Γ' be the graph obtained by removing e , $g = g(\Gamma')$,

and $\Gamma' = p_g[\Gamma'']$. We have $c_{\Gamma''} \in \Lambda$ and

$$c_{\Gamma}[S] = S p_{m(e)}[S] p_g \left[\frac{c_{\Gamma''}[S]}{S} \right] = S \frac{p_{m(e)}[S]}{p_g[S]} p_g[c_{\Gamma''}[S]].$$

This belongs to $S \Lambda \subset \Lambda$ because $g|m(e)$ by admissibility of Γ . \square

Moreover, the proof gives us the following statement:

Corollary 5.4 (of the proof). *We have the following explicit formula for $L^S[A, B]$:*

$$L^S[A, B] = \sum_{\substack{\Gamma \text{ admissible, } n=1 \\ \text{connected}}} \sum_{n=1}^{\infty} \phi_{b(\Gamma)}(g(\Gamma)) \frac{w(\Gamma)(A, B)}{\# \text{Aut}(\Gamma)} c_{\Gamma}[S],$$

where for each $\Gamma = (V, E, m)$ with $\phi_{b(\Gamma)}(g(\Gamma))$ we have

$$c_{\Gamma}[S] = S \frac{\prod_{e \in E} p_{m(e)}[S]}{\prod_{v \in V} p_{m(v)}[S]} \in S^{b(\Gamma)} \Lambda.$$

Similarly, we can employ the second convolution formula, in this case to compute the trace of an operator.

Definition 5.3. An S -admissible operator U is said to be of *trace class* if the sequence of the coefficients of $L_U^S[X, Y]$ is well-behaved. In this case we define its trace by

$$\text{Tr } U := \int_X^S K_U^S[X, X^*].$$

We have

Theorem 5.5. *The trace of an S -admissible operator U of trace class is admissible.*

6. ADMISSIBILITY OF Δ AND ∇

Now we set $S = -(1-q)(1-t)$, $\Lambda = \mathbb{Q}[q, t]$. The modified Macdonald polynomials $\{\tilde{H}_{\lambda}\}_{\lambda \in \mathcal{P}}$ form a basis of $\mathbb{Q}(q, t) \otimes \text{Sym}[X]$. For each $F \in \Lambda \otimes \text{Sym}[X]$ define $\Delta_F : \Lambda \otimes \text{Sym}[X] \rightarrow \mathbb{Q}(q, t) \otimes \text{Sym}[X]$ in the Macdonald basis as follows:

$$\Delta_F \tilde{H}_{\lambda} = F[B_{\lambda}], \quad B_{\lambda} = \sum_{c, r \in \lambda} q^c t^r,$$

the summation is over the cells (c, r) of λ , where c is the column index and r is the row index.

Over the slightly bigger ring $\mathbb{Q}(q, t)[[u]]$ define the operator

$$\Delta_u = \sum_{n=0}^{\infty} (-u)^n \Delta_{e_n}.$$

Then we have

$$\Delta_u^{-1} = \sum_{n=0}^{\infty} u^n \Delta_{h_n}.$$

Notice that

$$\Delta_u \tilde{H}_\lambda = \prod_{r,c \in \lambda} (1 - uq^c t^r) \tilde{H}_\lambda \quad (\lambda \in \mathcal{P}).$$

The top degree term of Δ_u in u is denoted by ∇ (see [BGHT99] for an overview of results about this operator, and note that our ∇ is different from the original one by a sign $(-1)^{|\lambda|}$),

$$(10) \quad \nabla \tilde{H}_\lambda = (-1)^{|\lambda|} q^{n'(\lambda)} t^{n(\lambda)} \tilde{H}_\lambda.$$

We also need the shift operators τ_u :

$$(\tau_u F)[X] = F[X + u], \quad \tau := \tau_1,$$

and their S -conjugates

$$(\tau_u^* F)[X] = F[X] \operatorname{Exp} \left[u \frac{X}{S} \right], \quad \tau^* := \tau_1^*.$$

Also we have the following partially defined operation on operators. For any continuous operator U denote by $\mathcal{S}^{-1}(U)$, if it exists, the unique continuous operator satisfying

$$\tau^* \tau U = \mathcal{S}^{-1}(U) \tau^* \tau.$$

Note that τ has its image in $\Lambda \otimes \operatorname{Sym}[X]$, τ^* is continuous, and both operators are S -admissible.

The following identity was established in [GM16]:

$$(11) \quad \Delta_v^{-1} \tau_u \Delta_v \tau_u^{-1} = \nabla^{-1} \tau_{uv} \nabla = \mathcal{S}^{-1}(\Delta'_{uv}),$$

where the operators act on $\Lambda[[u, v]] \otimes \operatorname{Sym}[X]$. This result was motivated by a conjecture in [BH13]. In [BH13] the identity (11) is shown to imply certain generalized Pieri rules for Macdonald polynomials. The main idea is that in the basis of Macdonald polynomials the operators Δ_v and ∇ are easily described, while the operators τ_u are difficult. So we write

$$\Delta_v^{-1} \tau_u \Delta_v = \nabla^{-1} \tau_{uv} \nabla \tau_u,$$

and then recursively express the result of a single application of the “difficult” τ -operator on the left using double application of τ -operators on the right. Our situation is the opposite, because we want to understand Log of the kernel of our

operators, so the τ -operators are “easy”, while the ∇ and Δ -operators are “difficult”. Thus we employ the other implication of (11):

$$\begin{aligned} \tau_u \Delta_v \tau_u^{-1} &= \Delta_v \mathcal{S}^{-1}(\Delta'_{uv}), \quad \text{equivalent to} \\ (12) \quad \tau_u \Delta_v \tau_u^{-1} \tau^* \tau &= \Delta_v \tau^* \tau \Delta_{uv} \text{Exp} \left[-\frac{uv}{S} \right]. \end{aligned}$$

Let $L_v[X, Y] = L_{\Delta_v}^S[X, Y] \in \mathbb{Q}(q, t)[[v]] \otimes \text{Sym}[[X, Y]]^*$. We compute the kernels of both sides of (12):

$$\begin{aligned} \tau_u \Delta_v \tau_u^{-1} \tau^* \tau \text{Exp} \left[\frac{XY}{S} \right] &= \tau_u \Delta_v \text{Exp} \left[\frac{(X-u)(Y+1)+Y}{S} \right] \\ &= \tau_u \text{Exp} \left[\frac{L_v[X, Y+1] + Y(1-u) - u}{S} \right] \\ &= \text{Exp} \left[\frac{L_v[X+u, Y+1] + Y(1-u) - u}{S} \right], \\ \Delta_v \tau^* \tau \Delta_{uv} \text{Exp} \left[\frac{XY}{S} \right] &= \Delta_v \tau^* \tau \text{Exp} \left[\frac{L_{uv}[X, Y]}{S} \right] \\ &= \Delta_v \tau^* \text{Exp} \left[\frac{L_{uv}[X+1, Y]}{S} \right] \\ &= \left(\text{Exp} \left[\frac{L_v[X, Z+1]}{S} \right], \text{Exp} \left[\frac{L_{uv}[Z+1, Y]}{S} \right] \right)_Z^S, \end{aligned}$$

where in the last formula we used the kernel for the operator $\Delta_v \tau^*$.

Define elements $A_{v,\lambda} \in \mathbb{Q}(q, t)[[v]] \otimes \text{Sym}[[X]]$ for $\lambda \in \mathcal{P}^*$ by the formula

$$L_v[X, Z+1] = \sum_{\lambda \in \mathcal{P}^*} A_{v,\lambda}[X] p_\lambda[Z].$$

Then (12) and the kernel evaluations above imply:

$$L_v[X+u, Y+1] + Y(1-u) - u = L^S(A_v[X], A_{uv}[Y]) - uv,$$

for L^S as in Corollary 5.4. Equivalently,

$$(13) \quad L_v[X+u, Y+1] = Y(u-1) + u(1-v) + L^S(A_v[X], A_{uv}[Y]).$$

Since Δ_v is a degree-preserving operator, we have an expansion

$$L_v[X, Y] = \sum_{k=1}^{\infty} L_v^{(k)}[X, Y],$$

where L_k has degrees k both in X and in Y . Denote by $T_k[X, Y; u, v]$ the sum of the terms on the right hand side of (13) with

$$(\text{degree in } X + \text{degree in } u) = k, \quad \text{degree in } u \geq 1, \text{ degree in } Y \leq k - 1.$$

Clearly these terms also have the degree in $X \leq k - 1$, therefore T_k can be computed from $L_v^{(i)}$ with $i \leq k - 1$, using only terms of Corollary 5.4 with $|\Gamma| \leq k - 1$. Note also, that the degree in v is bounded by the degree in X . On the other hand, we have

$$T_k[X, Y; 1, v] = L_v'^{(k)}[X + 1, Y] - L_v'^{(k)}[X, Y],$$

where

$$L_v'^{(k)}[X, Y] = L_v^{(k)}[X, Y + 1] - L_v^{(k)}[X, Y].$$

Notice that $L_v^{(k)}[X, Y]$ is homogeneous of degree k in Y , and $L_v'^{(k)}[X, Y]$ is still homogeneous of degree k in X . By the following well-known fact we can recover $L_v'^{(k)}[X, Y]$ from $T_k[X, Y; 1, v]$, and then $L_v^{(k)}[X, Y]$ from $L_v'^{(k)}[X, Y]$:

Lemma 6.1 (Solving a symmetric recursion). *For any $k \geq 1$ the map from symmetric functions of degree k to symmetric functions of degree $< k$ given by*

$$F[X] \rightarrow F'[X] := F[X + 1] - F[X]$$

is injective.

Proof. In the monomial basis, for $\lambda \in \mathcal{P}_k$ we have

$$m_\lambda[X + 1] - m_\lambda[X] = \sum_{j=1}^r m_{\lambda_1, \lambda_2, \dots, \widehat{\lambda_{i_j}}, \dots, \lambda_r}[X],$$

where i_1, i_2, \dots, i_r are such that the sequence $\lambda_{i_1}, \lambda_{i_2}, \dots, \lambda_{i_r}$ contains each element of the set of parts of λ exactly once. Thus, for instance, the map sending m_λ to $m_{\lambda'}$ with $\lambda' = (k - |\lambda|, \lambda_1, \dots, \lambda_{l(\lambda)})$ when $k - |\lambda| \geq \lambda_1$, and to 0 otherwise is a left inverse for the map $F \rightarrow F'$. \square

Thus we see that we can recursively compute each $L_v^{(k)}$ using only operations in the ring $\Lambda = \mathbb{Q}[q, t, v]$, which implies

Theorem 6.2. *The operator Δ_v is S -admissible over $\mathbb{Q}[q, t, v]$.*

The coefficients of L_{∇}^S can be extracted from the coefficients of $L_{\Delta_v}^S$. They are simply given by the terms of $L_{\Delta_v}^S$ whose degree in v equals to the degree in X , and hence also equals to the degree in Y . So we have

Corollary 6.3. *The operator ∇ is S -admissible over $\mathbb{Q}[q, t]$.*

Finally, the inverse of ∇ is related to ∇ by the identity

$$\bar{\omega}\nabla^{-1} = \nabla\bar{\omega},$$

where $\bar{\omega}$ is the λ -ring automorphism of $\mathbb{Q}[q, t, q^{-1}, t^{-1}] \otimes \text{Sym}[X]$ which sends q, t, X to $q^{-1}, t^{-1}, -X$. Thus we also have

Corollary 6.4. *The operator ∇^{-1} is S -admissible over the ring $\mathbb{Q}[q, t, q^{-1}, t^{-1}]$.*

In fact, the kernel of ∇^{-1} can be computed from the one of ∇ as follows:

$$L_{\nabla^{-1}}^S[X, Y; q, t] = qt L_{\nabla}^S[-X, -(qt)^{-1}Y; q^{-1}, t^{-1}].$$

6.1. Example of computation. We give the first few steps of the computation. So we start with $k = 1$ and the initial approximation $L_v[X, Y] \approx 0$. This gives us the first value

$$T_1[X, Y, u, v] = u(1 - v).$$

Solving the symmetric recursion for $k = 1$, $T_1[X, Y, 1, v] = (1 - v)$ gives

$$L_v^{(1)}[X, Y] = XY(1 - v).$$

Now we proceed to $k = 2$, $L_v[X, Y] \approx XY(1 - v)$. We need to compute the operation L^S for

$$L_v[X, Z + 1] \approx X(Z + 1)(1 - v) = X(1 - v) + X(1 - v)Z,$$

$$L_{uv}[Y, Z + 1] \approx Y(1 - uv) + Y(1 - uv)Z.$$

The graphs of degrees 0 and 1 (see Section 3.5) produce

$$X(1 - v) + Y(1 - uv) + X(1 - v)Y(1 - uv),$$

which together with the extra summand $Y(u - 1) + u(1 - v)$ from (13) gives

$$(1 - v)((X + u)(Y + 1) - uvXY).$$

Keeping only the terms of correct degrees we obtain

$$T_2[X, Y; u, v] = uv(v - 1)XY,$$

and the solution to the symmetric recursion is

$$L_v^{(2)}[X, Y] = v(v - 1)e_2[X]e_2[Y].$$

We will compute one more step $k = 3$. Expansion of $L_v[X, Z + 1]$ in the power sum basis is

$$L_v[X, Z + 1] \approx X(1 - v) + X(1 - v)Z + v(v - 1)e_2[X] \left(\frac{Z^2 - p_2[Z]}{2} + p_1[Z] \right),$$

$$L_{uv}[Y, Z + 1] \approx Y(1 - uv) + Y(1 - uv)Z + uv(uv - 1)e_2[Y] \left(\frac{Z^2 - p_2[Z]}{2} + p_1[Z] \right).$$

So the logarithmic convolution is

$$\begin{aligned} & X(1 - v) + Y(1 - uv) + (X(1 - v) + v(v - 1)e_2[X]) (Y(1 - uv) + uv(uv - 1)e_2[Y]) \\ & \quad - \frac{1}{2}v(v - 1)e_2[X]uv(uv - 1)e_2[Y](q + 1)(t + 1) - \frac{1}{2}v(v - 1)e_2[X]p_2[Y(1 - uv)] \\ & \quad - \frac{1}{2}p_2[X(1 - v)]uv(uv - 1)e_2[Y] - \frac{1}{2}v(v - 1)e_2[X]uv(uv - 1)e_2[Y](q - 1)(t - 1) \\ & \quad + \frac{1}{2}v(v - 1)e_2[X](Y(1 - uv))^2 + \frac{1}{2}uv(uv - 1)e_2[Y](X(1 - v))^2, \end{aligned}$$

where we kept only terms with degrees in X and Y not exceeding 2. Keeping only terms with correct degrees and setting $u = 1$ we obtain

$$\begin{aligned} T_2[X, Y; 1, v] &= v(v - 1)(1 - v(q + t))e_2[X]e_2[Y] - v^2(v - 1)(e_2[Y]X^2 + e_2[X]Y^2) \\ &\quad - v^2(v - 1)(Xe_2[Y] + Ye_2[X]). \end{aligned}$$

Next, we pass to the monomial basis in order to apply the procedure from the proof of Lemma 6.1. We throw away the terms $m_{(1,1)}[X]m_{(2)}[Y]$, because they will go to 0 anyway. We are left with

$$v(v - 1)(1 - v(q + t + 4))m_{(1,1)}[X]m_{(1,1)}[Y] - v^2(v - 1)(Xm_{(1,1)}[Y] + Ym_{(1,1)}[X]),$$

So we find

$$\begin{aligned} L_v^{(3)} &= v(v - 1)(1 - v(q + t + 4)) m_{(1,1,1)}[X] m_{(1,1,1)}[Y] \\ &\quad - v^2(v - 1) (m_{(2,1)}[X] m_{(1,1,1)}[Y] + m_{(1,1,1)}[X] m_{(2,1)}[Y]). \end{aligned}$$

Note that replacing v by $-v$ makes all the coefficients positive. This is expected to hold in general.

7. HLK KERNELS

Fix integers $g, n \geq 0$. In what follows, the variables q, t, T, u_i are monomial, i.e $p_k[q] = q^k$ for all k , and similarly for t, T and u_i for all i . The exponential HLK

kernel of genus g with n punctures (see [HLRV11], [CRV16]) is defined as

$$(14) \quad \Omega_{u_1, u_2, \dots, u_g}[X_1, X_2, \dots, X_n; q, t, T] = \sum_{\lambda \in \mathcal{P}} \frac{\prod_{i=1}^n \tilde{H}_\lambda[X_i; q, t] \prod_{i=1}^g N_\lambda(u_i; q, t)}{(\tilde{H}_\lambda, \tilde{H}_\lambda)^S} T^{|\lambda|},$$

where

$$N_\lambda(u; q, t) = (-u)^{-|\lambda|} q^{n(\lambda')} t^{n(\lambda)} \prod_{s \in \lambda} (1 - uq^{-a(s)} t^{l(s)+1}) (1 - ut^{-l(s)} q^{a(s)+1}),$$

the variable s runs over the cells of λ , and $a(s)$, $l(s)$ denote the arm and the leg lengths of s correspondingly. Note that for $n = 0$ the variable T is necessary for convergence. For $n > 0$ we can set $T = 1$ without losing any information: the power of T is always equal to the degree in any of the variables X_1, X_2, \dots . The kernel belongs to the Λ -ring

$$\mathbb{Q}(q, t)[u_1, u_2, \dots, u_g, u_1^{-1}, u_2^{-1}, \dots, u_g^{-1}][[T]] \otimes \text{Sym}[[X_1, X_2, \dots, X_n]].$$

In [CRV16] a more convenient formula for N_λ is given. We sketch a proof of this formula. It is convenient to use the notation (remember $S = -(1 - q)(1 - t)$)

$$D_\lambda(q, t) = -1 - SB_\lambda, \quad \bar{D}_\lambda(q, t) = D_\lambda(q^{-1}, t^{-1}).$$

Then it is not hard to prove that:

$$qt \frac{D_\lambda \bar{D}_\lambda - 1}{S} = \sum_{s \in \lambda} q^{-a(s)} t^{l(s)+1} + t^{-l(s)} q^{a(s)+1},$$

which implies

$$N_\lambda(u; q, t) = (-u)^{-|\lambda|} q^{n(\lambda')} t^{n(\lambda)} \text{Exp} \left[-qt \frac{D_\lambda \bar{D}_\lambda - 1}{S} u \right] \in \mathbb{Q}[q, t, q^{-1}, t^{-1}]((u)).$$

We will use Tesler's operator (see [GHT99b], [Mel16]), defined as $\nabla \tau^* \tau$, whose main property is:

$$(15) \quad \nabla \tau^* \tau \tilde{H}_\lambda = \text{Exp} \left[\frac{D_\lambda X}{S} \right].$$

Applying the operator $\bar{\omega}$ on both sides we obtain

$$\nabla^{-1} \tau_{-qt}^* \tau_{-1} \nabla^{-1} \tilde{H}_\lambda = \text{Exp} \left[-qt \frac{\bar{D}_\lambda X}{S} \right].$$

We substitute uX in the place of X in the former expression, and take the S -modified scalar product with the latter expression:

$$\left(\nabla \tau_u^* (\tilde{H}_\lambda[uX + 1]), \nabla^{-1} \tau_{-qt}^* \tau_{-1} \nabla^{-1} \tilde{H}_\lambda \right)^S = \text{Exp} \left[-qtu \frac{D_\lambda \bar{D}_\lambda}{S} \right].$$

Now we use the definition of ∇ from (10) and the fact that the degree of \tilde{H}_λ is $|\lambda|$:

$$N_\lambda(u) = \left(\nabla \tau_u^* \tau_{u^{-1}} \tilde{H}_\lambda, \nabla^{-1} \tau_{-qt}^* \tau_{-1} \tilde{H}_\lambda \right)^S \text{Exp} \left[\frac{uqt}{S} \right].$$

Since ∇ is self-adjoint with respect to the S -scalar product, we can remove it from both sides:

$$N_\lambda(u) = \left(\tau_u^* \tau_{u^{-1}} \tilde{H}_\lambda, \tau_{-qt}^* \tau_{-1} \tilde{H}_\lambda \right)^S \text{Exp} \left[\frac{uqt}{S} \right].$$

Finally, we move τ_u^* to the right, where it becomes τ_u , and pass it through $(\tau_{qt}^*)^{-1}$, which gives an extra factor of $\text{Exp} \left[-\frac{uqt}{S} \right]$. Then we move $(\tau_{qt}^*)^{-1}$ to the left:

$$(16) \quad N_\lambda(u) = \left(\tau_{u^{-1}-qt} \tilde{H}_\lambda, \tau_{u^{-1}} \tilde{H}_\lambda \right)^S.$$

The identity is understood in the ring $\mathbb{Q}[q, t][u, u^{-1}]$. We are ready to prove our main result

Theorem 7.1. *For any $n, g \geq 0$ the exponential HLV kernel of genus g with n punctures is S -admissible over the ring*

$$\Lambda = \mathbb{Q}[q, t, u_1, u_2, \dots, u_g, u_1^{-1}, u_2^{-1}, \dots, u_g^{-1}][[T]] \otimes \text{Sym}[[X_1, X_2, \dots, X_n]]$$

with $S = -(q-1)(t-1)$. Equivalently, the logarithmic HLV kernel

$$\begin{aligned} & \mathbb{H}_{u_1, u_2, \dots, u_g}[X_1, X_2, \dots, X_n; q, t, T] \\ & := -(q-1)(t-1) \text{Log } \Omega_{u_1, u_2, \dots, u_g}[X_1, X_2, \dots, X_n; q, t, T] \end{aligned}$$

is in Λ .

Proof. Each coefficient of Ω in the variables u_i , T and X_i , as a function of q and t belongs to the intersection of the rings³

$$\mathbb{Q}(q, t) \cap \mathbb{Q}[q, q^{-1}][[t]] \cap \mathbb{Q}[t, t^{-1}][[q]].$$

This follows from invertibility of the denominator in (14), explicitly given as

$$(\tilde{H}_\lambda, \tilde{H}_\lambda)^S = \prod_{s \in \lambda} (q^{a(s)} - t^{l(s)+1}) (q^{a(s)+1} - t^{l(s)})$$

in these three rings. Thus it is enough to show that Ω is admissible over

$$\Lambda' = \mathbb{Q}[q, t, q^{-1}, t^{-1}, u_1, u_2, \dots, u_g, u_1^{-1}, u_2^{-1}, \dots, u_g^{-1}][[T]] \otimes \text{Sym}[[X_1, X_2, \dots, X_n]].$$

Indeed, we have

$$\mathbb{Q}(q, t) \cap \mathbb{Q}[q, q^{-1}][[t]] \cap \mathbb{Q}[t, t^{-1}][[q]] \cap \mathbb{Q}[q, t, q^{-1}, t^{-1}] = \mathbb{Q}[q, t].$$

³The intersection is understood inside the ring $\mathbb{Q}(q, t)$

The next step is to show how to reconstruct Ω using the operators ∇ , ∇^{-1} , etc, which are S -admissible by Corollaries 6.3 and 6.4. Begin with the kernel of Tesler's operator $\nabla\tau^*\tau$, which is admissible by Corollary 6.3. Applying $\nabla\tau^*\tau$ to the Cauchy kernel

$$\text{Exp} \left[\frac{XY}{S} \right] = \sum_{\lambda \in \mathcal{P}} \frac{\tilde{H}_\lambda[X] \tilde{H}_\lambda[Y]}{(\tilde{H}_\lambda, \tilde{H}_\lambda)^S}$$

we obtain

$$\sum_{\lambda \in \mathcal{P}} \frac{\text{Exp} \left[\frac{XD_\Delta}{S} \right] \tilde{H}_\lambda[Y]}{(\tilde{H}_\lambda, \tilde{H}_\lambda)^S}.$$

So this expression is admissible. Now we perform the substitution $X = X_1 + X_2 + \dots + X_{n+2g}$, $Y = T$. This does not affect admissibility and the result is

$$\sum_{\lambda \in \mathcal{P}} \frac{\prod_{i=1}^{n+2g} \text{Exp} \left[\frac{X_i D_\Delta}{S} \right] T^{|\lambda|}}{(\tilde{H}_\lambda, \tilde{H}_\lambda)^S}.$$

Next we apply $\tau_{-1}^* \nabla^{-1}$. Note that ∇^{-1} is not admissible over Λ , but is still admissible over Λ' . By Lemma 5.3 applying ∇^{-1} does not affect admissibility, and by (15) we obtain

$$\sum_{\lambda \in \mathcal{P}} \frac{\prod_{i=1}^{n+2g} \tilde{H}_\lambda[X_i + 1] T^{|\lambda|}}{(\tilde{H}_\lambda, \tilde{H}_\lambda)^S}.$$

Next we throw away all the terms whose degree in any of X_1, X_2, \dots is less than the degree in T . This produces

$$\sum_{\lambda \in \mathcal{P}} \frac{\prod_{i=1}^{n+2g} \tilde{H}_\lambda[X_i] T^{|\lambda|}}{(\tilde{H}_\lambda, \tilde{H}_\lambda)^S},$$

precisely the kernel for $n+2g$ punctures. Next we apply $\tau_{u_i^{-1}-qt}$ in X_{n+2i-1} and τ_{u_i-1} in X_{n+2i} for each $i = 1, 2, \dots, g$. Then we view the resulting expression as a kernel of an operator with $X = X_{n+2i-1}$, $Y = X_{n+2i}$ for $i = 1, 2, \dots, g$, and take the trace. By Theorem 5.5, the result is still admissible. We obtain

$$\sum_{\lambda \in \mathcal{P}} \frac{\prod_{i=1}^n \tilde{H}_\lambda[X_i] \prod_{i=1}^g (\tilde{H}_\lambda[X + u_i^{-1} - qt], \tilde{H}_\lambda[X + u_i - 1])_X^S T^{|\lambda|}}{(\tilde{H}_\lambda, \tilde{H}_\lambda)^S},$$

which by (16) equals $\Omega_{u_1, u_2, \dots, u_g}[X_1, X_2, \dots, X_n]$. \square

The coefficients of the expansion of $\Omega_{u_1, u_2, \dots, u_g}[X_1, X_2, \dots, X_n]$ in the monomial basis, as functions in q and t are in $\mathbb{Z}[q, q^{-1}][[t]]$, which follows from integrality of Macdonald polynomials. It is well-known that Log , when computed in the monomial basis has coefficients in \mathbb{Z} . This implies that the coefficients of \mathbb{H} in the monomial basis are in $\mathbb{Z}[q, q^{-1}][[t]]$, so by our result the coefficients are in $\mathbb{Z}[q, t]$.

Corollary 7.2. *The coefficients of*

$$\mathbb{H}_{u_1, u_2, \dots, u_g}[X_1, X_2, \dots, X_n; q, t, T]$$

in the monomial basis are polynomials in q, t, u_i, u_i^{-1} with integer coefficients.

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